MTH 311

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Advanced Calculus I Workbook

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Contents

1 Introduction

2 The Field of Rational Numbers

Shortly after attaining mastery over counting numbers, it is not uncommon in schools for students to hear the word "fraction" for the first time. The concept of what there truly is to a fraction however, would likely come as a surprise to most. The fraction strikes at the heart of what many would consider natural, simple, and elementary. Yet at the same time, it would be a gross understatement to trivialize the concept of the fraction with these labels alone. For mathematicians and non-mathematicians alike, it is practically impossible to refute the change that fractions have delivered to our human society. In truth, fractions and their fascinating properties boil down to a lot more than the decision to divide two random numbers.

Fractions bear more subtlety than they are often given credit for the first time people learn them. In mathematics, we construct fractions as ordered pairs of (a, b) where $b \neq 0$ such that $a, b \in \mathbb{Z}$ and we write it as $\frac{a}{b}$. While simple in construction, the idea of what truly constitutes a fraction, as a consequence of this definition, is actually quite broad. To start, it is more appropriate to regard fractions as organized sets of the same value than to think of fractions as disjoint division problems. While in elementary and middle school, we are generally taught $\frac{1}{2} = \frac{2}{4}$ and given mathematical justification for it. However we are often robbed of the larger picture, which is, that the equality of rational numbers is ultimately an equivalence relation.

In mathematics, we say that a relation R on a nonempty set is an equivalence relation if R is reflexive, symmetric, and transitive[1]. Proving that the set of all fractions form equivalence relations is quite simple when we define the equality of fractions through the following binary relation:

$$\frac{a}{b} \equiv \frac{c}{d}$$

where $b, d \neq 0$ and $a, b, c, d \in \mathbb{Z}$. What is important to take away from this is relation is that $\frac{a}{b} \equiv \frac{c}{d}$ means that $ad = bc$ as integers. Ultimately, this enables us to show that for every fraction that is in reduced form, there exists a unique subset of the set of all fractions which contains all equivalent fractions in it. For example, let $\mathrm{O} = \{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, ...\}$, $r \in \mathbb{Z}$ Taking any two fractions from O and cross multiplying them will tell you that the LHS and RHS are always equal. Collectively, the rational numbers, which we denote $\mathbb{Q}$, as a whole can be partitioned into an infinite amount of subsets that behave in this fashion. This is the equivalence relation that all rational numbers follow.

Exercise: Prove the following properties of the binary relation $\equiv$:

Reflexive: Assume that you have a fraction $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. By rules of the construction of fractions, $\frac{a}{b}$ can be re-expressed as an ordered pair (a, b). Thus:

$$P_1 = (a, b) = \frac{a}{b}$$

for $P_1 \in \mathbb{Z}^2$. The reflexive property is defined as follows:

$$P_1 R P_1$$

$$(a, b)R(a, b)$$

$$ab = ba$$

Therefore the relation is reflexive.

Symmetric: Assume that you have two fractions $\frac{a}{b}$ and $\frac{c}{d}$ related by the relation R such that $\frac{a}{b} \equiv \frac{c}{d}$ where $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$. By the construction rules of fractions, fractions can be represented as ordered pairs. Thus:

$$P_1 = (a, b) = \frac{a}{b}$$

$$P_2 = (c, d) = \frac{c}{d}$$

for $P_1, P_2 \in \mathbb{Z}^2$. The symmetric property is defined as follows:

$$P_1 R P_2 \rightarrow P_2 R P_1$$

Working on both sides of the conditional you can see that:

$$P_1 R P_2 \rightarrow P_2 R P_1$$

$$(a, b)R(c, d) \rightarrow (c, d)R(a, b)$$

$$ad = cb \rightarrow cb = ad$$

Since the relation is equal on both sides, the relation is symmetric.

Transitive: Assume you have three fractions $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ under a relation R such that $\frac{a}{b} \equiv \frac{c}{d}$ and $\frac{c}{d} \equiv \frac{e}{f}$where $a, b, c, d, e, f \in \mathbb{Z}$ and $b, d, f \neq 0$. By construction rules of fractions, each individual fraction can be be represented as an ordered pair. That is, let:

$$P_1 = (a, b) = \frac{a}{b}$$

$$P_2 = (c, d) = \frac{c}{d}$$

$$P_3 = (e, f) = \frac{e}{f}$$

for $P_1, P_2, P_3 \in \mathbb{Z}^2$. The transitive property is defined as follows:

$$P_1 R P_2, P_2 R P_3 \rightarrow P_1 R P_3$$

Addressing the relation on both sides of the conditional yields the following result:

$$P_1 R P_2, P_2 R P_3 \rightarrow P_1 R P_3$$

$$(a, b)R(c, d), (c, d)R(e, f) \rightarrow (a, b)R(e, f)$$

$$ad = cb, cf = ed \rightarrow af = eb$$

If you solve for a and f individually you get the following:

$$ad = cb$$

$$a = \frac{cb}{d}$$

$$cf = ed$$

$$f = \frac{ed}{c}$$

Then substitute in a and f on the right side of the conditional:

$$af = eb$$

$$\frac{cb}{d} * \frac{ed}{c} = eb$$

$$be = eb$$

Which is clearly true, meaning that the relation R is transitive. Since R is reflexive, symmetric, and transitive it is by definition an equivalence relation.

2.1 The Division algorithm

By this point, you should be well-informed on how much information can be drawn from the well of rational numbers. More importantly, it should be clear that the rational numbers are significantly different than the division problems we think they are when initially learning fractions in elementary and middle school. This subsection of the chapter has been included to provide a basic insight on how differently division and fractions differ in the study of mathematics.

The primary difference between division and fractions is the way in which they deal with excess quantities. As emphasized in the previous section, most people are originally taught fractions in a way that struggles to represent the rationals as they truly are. Fractions are often simplified as merely parts of a whole. While not entirely wrong, this notion is also taught to be, at best, conditionally true. There are many times in grade school where the concept of an improper or a mixed fraction comes up, where the value of the numerator exceeds the denominator. These may bring confusion and frustration to many at first, but ultimately they are just normal cases of normal fractions.

In contrast to the idea of an improper fraction, division is taught with the concept of a remainder. Any values left over after dividing which don't fit in evenly are moved to the side and labeled this.

At this point, one may question what the difference between improper fractions and division with a remainder at the end really is. In truth: nothing.

Dividing anything with a remainder left over at the end is in mathematics, exactly the same as writing a large number over a smaller number with a line in between. The only contrast is the way in which they are represented. However, there is great value one can gain from learning two sides of the same coin. After this chapter, you should have more of an understanding on how division and fractions have powerful ways of representing the same phenomenon.

In mathematics, we say that a $a \in \mathbb{Z}$ is called a multiple of $b \in \mathbb{Z}$ if $a = bq$ for some $q \in \mathbb{Z}$. We also say that b is a divisor(or factor) of a, and we write $b|a$ to denote this[2].

Having familiarized ourselves some basic terms and definitions, we are now ready to define the *Well-Ordering Principle*. Once we have done this, we will be able to make sense out of the division algorithm and prove it.

Forward: The Well-Ordering Principle is a foundation to numerous realms of mathematics. As an *axiom*, is it accepted true by definition and it cannot be proven.

Axiom: *The Well-Ordering Principle*
Every nonempty set of natural numbers contains a smallest element[2].

Now that we have defined the *Well-Ordering Principle*, we can claim and prove the *Division Algorithm*.

1. **The Division Algorithm:**
   For any $a, b \in \mathbb{Z}$, with $b > 0$, there exists a unique $q \in \mathbb{Z}$ called the quotient and a unique $r \in \mathbb{Z}$ called the remainder such that:

$$a = bq + r$$

   with $0 \leq r < b$.

2. **Proof:**

   (a) Begin by rearranging the formula of the Division Algorithm to solve for r:

$$a = bq + r$$

$$r = a - bq$$

   (b) Now consider the set $R = \{a - bq : a, b, q \in \mathbb{Z}\}$ where $b > 0$.

   (c) Consider the division algorithm and how we have rearranged it. With the work we have done so far, the elements in R represent the set of all potential remainders. Among these, we need to find the smallest $r \in R$ such that $0 \leq r < b$.

   (d) Let $R^+ =$ the set of non-negative integers in R. Also note that this implies that $R^+ \subseteq R$.

   (e) We are told that $b > 0$. Since we know that $b \in \mathbb{Z}$, this tells us that $b \geq 1$ must be true.

   (f) Since we know $b \geq 1$ must be true, consider what happens to r when you suppose $q = -|a|$:

$$r = a - bq$$

$$r = a - b(-|a|)$$

$$r = a + b|a|$$

   Notice that the final term is non-negative. This means that R^+ is nonempty and by the *Well-Ordering Principle*, R^+ has a smallest element $r \in R, R^+$.

   (g) By definition, $r \geq 0$ and since $r \in R^+$, it follows that:

$$r = a - bq$$

   for some $a, b, q \in \mathbb{Z}$ where $b > 0$.

   (h) We have already shown that $0 \leq r$. Now we must show that $r < b$. To do this, we will assume the negation of $r < b$ that $r \geq b$. Since $r \geq b$, we can let $t = r - b$ where $t \geq 0$ and where $t \in \mathbb{Z}$. Now observe what happens when we substitute r into our equation for t:

$$t = r - b$$

$$t = (a - bq) - b$$

$$t = a - b(q + 1)$$

   Notice that we let $t = r - b$ which implies that $t \geq 0$, where $t \in \mathbb{Z}$. But when we made the substitutions above, we showed that $t = a - bq - b$ and $t = a - bq - b$. This is a contradiction since this means $t < r$. This makes no sense since when we applied the *Well-Ordering Principle* we used it to say that r was the smallest element in R^+.

(i) Since $r \geq b$ is a contradiction, we must have $r < b$. Therefore, r and q must exist and $0 \leq r < b$. Lastly, the claim of the algorithm, that $a = bq + r$ must also be true.

(j) In order to finish our proof of the *Division Algorithm*, the only thing we need to do now is prove that q and r are unique.

(k) To show that q and r are unique assume that we can also write:

$$a = bp + s$$

where $p, s \in \mathbb{Z}$ with $0 \leq s < b$.
Now, take a close look at the ranges that both equations gave us:

$$0 \leq r < b$$

$$0 \leq s < b$$

If you look at this long enough, you'll notice that this means that:

$$|s - r| < b$$

must also be true.

(l) We know that:

$$a = a$$

$$bp + s = bq + r$$

and so we have the ability to re-arrange this so that:

$$bp + s = bq + r$$

$$s - r = b(q - p)$$

which implies that:

$$b|(s - r)$$

We just proved that $b > |s - r|$, so in order for $b|(s - r)$ then $s - r = 0$. This effectively means that:

$$s = r$$

$$s - r = b(q - p)$$

$$0 = b(q - p)$$

but we already know that $b > 0$. Therefore $b \neq 0$ and so $q = p$
Notice that the quotients and the remainders have to be the same regardless of which we chose. This means that they have the same solution, which must be same unique solution.

Q.E.D.

2.2 The Euclidean Algorithm

Understanding the basis of division is critical to understanding powerful methods that employ it. The Euclidean Algorithm is a shining star in the world of mathematics and in how efficiently it solves for the gcd of two integers. Originally found in the ancient works of Euclid himself, it is not surprising to hear that many mathematicians have spent time observing and drawing parallels to the method. In the words of D.E. Knuth, "[The Euclidean algorithm] is the granddaddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day[3]." The Euclidean Algorithm, which models how division is nothing more than repeated subtraction, is widely used in computer programs today.

The easiest way to describe the Euclidean example is through example. Essentially, given a pair of integers (a, b), we will always subtract the smaller from the larger. Once we have done this, we replace whatever is left with the previous larger term and repeat the process. This is repeated until we come to a zero, at which point, whatever term we have is the $gcd(a, b)$.

1. Use the Euclidean Algorithm to find the $gcd(35, 14)$:

$$gcd(35, 14)$$
$$35 - 14 = 21$$
$$gcd(21, 14)$$
$$21 - 14 = 7$$
$$gcd(14, 7)$$
$$14 - 7 = 7$$
$$gcd(7, 7)$$
$$7 - 7 = 0 \longrightarrow gcd(35, 14) = 7$$

Using matrices to carry through the same computation can also be worthwhile:

2. $$\begin{bmatrix} 1 & 0 & 35 \\ 0 & 1 & 14 \end{bmatrix} \underset{R_1 - 2R_2}{\longrightarrow} \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & 14 \end{bmatrix} \underset{R_2 - 2R_1}{\longrightarrow} \begin{bmatrix} 1 & -2 & 7 \\ -2 & 5 & 0 \end{bmatrix}$$

Therefore $gcd(35, 14) = 7$.

Notice that when we choose to use matrices, we are essentially creating a set of linear combinations $ma + nb$ that will always satisfy the conditions of the matrix. In other words:

$$(1)(35) + (0)(14) = 35$$
$$(0)(35) + (1)(14) = 14$$
$$(1)(35) + (-2)(14) = 7$$
$$(0)(35) + (1)(14) = 14$$
$$(1)(35) + (-2)(14) = 7$$

$$(-2)(35) + (5)(14) = 0$$

are all true. Furthermore, since this linear combination can be represented on matrices, the number of linear combinations for values of $m, n \in \mathbb{Z}$ is infinite since the possible matrix operations that can be applied are infinite. The Euclidean Algorithm, while simple in process is broad in its extent. Once we have clarified terminology on the nature of reduced fractions, we will observe a model known as the *Stern-Brocot Tree* and emphasize the ways in which it parallels the Euclidean Algorithm.

2.3 Reduced Fractions

Up until this point, we have discussed the basics of rational numbers, the equivalence relation that spans them, the power of the Well-Ordering Principle, and some fundamental algorithms in the realm of rational numbers and division. However, there is still more emphasis to be placed on the meaning of what we call a *reduced fraction*.

By definition, A fraction of the form $\frac{a}{b}$ is called a *reduced fraction* if the ordered pair $(a, b) \in \mathbb{Z}$ is *relatively-prime*. Any two numbers $a, b \in \mathbb{Z}$ are *relatively-prime* if they only have 1 as a common factor. Lastly, we say that a fraction $\frac{a}{b}$ is in *canonical form* if $\frac{a}{b}$ is a reduced fraction where $b > 0$.

Exercise: Prove that in each equivalence class for $\equiv$ there is a reduced fraction $\frac{a}{b}$.

Re-writing the statement we are trying to prove with mathematical quantifiers helps to analyze the logic behind it. Substituting in the quantifiers and a definition of a reduced fraction yields the following statement:

P: Prove that for $\forall$ equivalence classes for $\equiv$, $\exists \frac{a}{b} \in \mathbb{Q}$ such that $gcd(a, b) = 1$.

Via method of a proof by contradiction, we assume $\neg P$, that is:

¬P: $\exists$ equivalence class for $\equiv$ such that $\forall \frac{a}{b} \in \mathbb{Q}$, $gcd(a, b) \neq 1$.

Assume ¬P and that you have an equivalence class of fractions for $\equiv$ such that $\forall \frac{a}{b} \in \mathbb{Q}$ but $gcd(a, b) \neq 1$. This means that you can choose an arbitrary fraction $\frac{a}{b} \in \mathbb{Q}$ within this equivalence class that does not reduce down so that a and b are relatively prime. However, if a fraction $\frac{a}{b}$ cannot be reduced, then that fraction *must* be relatively-prime and $gcd(a, b) = 1$ must also be true. Contradiction! Assuming ¬P leads to a contradiction, therefore P is correct and we are done.

Exercise: Prove that if $\frac{a}{b}$ and $\frac{c}{d}$ are in canonical form and $\frac{a}{b} \equiv \frac{c}{d}$ then $a = c$ and $b = d$.

(a) If $\frac{a}{b}$ and $\frac{c}{d}$ are in canonical form then $gcd(a, b)$ and $gcd(c, d)$ are both equal to 1 (since they must be reduced fractions). Earlier on, we proved that $\equiv$ was an equivalence relation. We also stressed how $\frac{a}{b} \equiv \frac{c}{d}$ meant that $ad = bc$ where $a, b, c, d \in \mathbb{Z}$ while proving the equivalence relation.

(b) Recall the matrix example mentioned above for the Euclidean Algorithm. It follows from the logic of the Euclidean Algorithm that we can represent $gcd(a, b)$ as a row of a matrix that forms a linear combination with $m, n \in \mathbb{Z}$. We can also do this to represent

$gcd(c,d)$ via a matrix that forms a linear combination with $p,q \in \mathbb{Z}$.

$$\begin{bmatrix} m & n & 1 \end{bmatrix} \Rightarrow ma + nb = 1$$
$$\begin{bmatrix} p & q & 1 \end{bmatrix} \Rightarrow pc + qd = 1$$

(c) Since $\frac{a}{b} \equiv \frac{c}{d}$ meant that $ad = bc$, we can use algebra to solve the following:

$$ad = bc$$
$$a = \frac{bc}{d}$$

We already established that $gcd(c,d) = 1$. If $a = \frac{bc}{d}$ then this really implies that d divides b. This is because if the $gcd(c,d) = 1$ then c cannot be a divisor of d and d cannot be a divisor of c, unless one or both of them are equal to 1.

REVISIT THIS LATER

Having shown a few different results, it is time to learn about yet another application of the rationals. Before we move on entirely, it is important to quickly define some other useful terms that apply to the rational numbers, so that the reader may have material to experiment and find results with on their own.

(a) Addition of fractions is defined as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

(b) Multiplication of fractions is defined as follows:

$$\frac{a}{b} * \frac{c}{d} = \frac{ac}{bd}$$

(c) Order is a concept in mathematics that is very useful to understand and use. For fractions, we write the binary relation $<$ on the set of rational numbers $\mathbb{Q}$ as follows:

For rational numbers $\frac{a}{b}, \frac{c}{d}$ that are in canonical form, $\frac{a}{b} < \frac{c}{d}$ means $ad < bc$.

2.4 Stern-Brocot Tree

One of the most interesting features in mathematics and its history is the independence with which ground-breaking math is often discovered. The most famous example of course, being the independent discovery of calculus by both Gottfried Liebniz and Isaac Newton and their fight for the credit of their ideas. Credit in mathematics has stirred controversy for hundreds of years. It was not uncommon for mathematicians to carry the secrets of their research to their grave. Sometimes mathematicians in fear of being forgotten contributed to their own downfall through the slander and accusations they made, such as Tartaglia, who as one of the pioneers behind the idea of the depressed cubic vehemently accused Cardan of plagiarizing him[4]. For every several thousand mathematical results, the likelihood of a Ramanujan coming along and independently proving them seems only more and more likely as time goes on. At any one time, hundreds of mathematicians could be working on the same problem completely isolated from each other, yet still producing the same results. Sometimes, pioneers of mathematics aren't even mathematicians–like how Fermat was a lawyer.

The Stern-Brocot Tree is an example of this independent phenomenon of discovery in mathematics. Attention to the Stern-Brocot tree was originally drawn in 1858 when Moritz Stern published a formal paper on the tree and its applications in number theory. Stern had succeeded Gauss's position at the University of Gottingen in Germany and taken the mathematician's approach to the model [5]. However, the tree was almost simultaneously discovered in 1861 by Achille Brocot, who had been born into a family of skilled clockmakers and who developed the tree as an aid for him to model gear ratios [6].

The Stern-Brocot is a binary tree where every entry is a fraction $\frac{a}{b}$ with two children α or β, where α,β are known as the left and right children respectively.

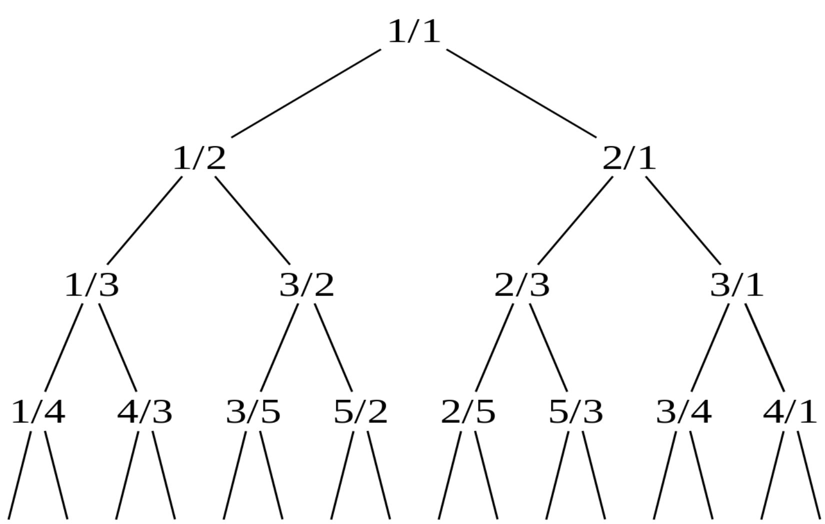
1. α is often referred to as the "left child" and defined as follows:

$$\alpha = \frac{a}{a+b}$$

2. β is often referred to as the "right child" and defined as follows:

$$\beta = \frac{a+b}{b}$$

Starting with $\frac{1}{1}$, we generate the following model famously known as the Stern-Brocot Tree:



Exercise: Prove the following:

(a) All the fractions in the Stern-Brocot Tree are in canonical form.

(b) Every positive rational number occurs once and only once in the Stern-Brocot tree.

(c) There is a one-to-one correspondence between the positive integers and the positive rational numbers.

In mathematics, a set is defined as *countable* if that set is finite or denumerable. A set P is *denumerable* if the cardiniality of P is equal to the cardinality of the natural numbers. That is, a set P is denumerable if $|P| = |\mathbb{N}|$. A one-to-one correspondence between two sets exists only if there is a bijection between the two sets. Therefore, in order to show that there is a one-to-one correspondence between the positive integers and the positive rational numbers, we need to prove that there is a bijection that can be written between the two sets[1].

i. Begin by recognizing that as the positive integers are a subset of the integers it follows that the positive integers are also denumerable.
ii. Observe the table below, and note that the diagonal of the table always reduces to one.

3 The Square Root of Two

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